Problem 1: A metal sphere of radius carries a charge of Q. It is surrounded, out to radius b, by linear dielectric material of permittivityϵ. Find the potential at the center of the sphere.

Sketch Problem

Start by telling me everything you can about the potential and how to solve the problem?

Use Gauss's Law:

$$\oint \vec{D} \cdot \overrightarrow{da} = Q_{f-enc}$$

Where are there free charges and bound charges?

Find D, E everywhere

Now find V everywhere

Now let's find the bound charge densities, remember:

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

$$\rho_b = -\vec{\nabla} \cdot \vec{P}$$

$$\sigma_b = \vec{P} \cdot \hat{n}$$

Example: A sphere of homogenous linear dielectric material is placed in an otherwise uniform electric field $E_o\hat{k}$. Find the electric field inside the sphere.

What are our Boundary Conditions:

At r → 0

$$V_{in} \to Finite$$

At the boundary

$$V_{in} = V_{out}$$

$$\epsilon_{in}\frac{\partial V_{in}}{\partial n} - \epsilon_{out}\frac{\partial V_{out}}{\partial n} = \sigma_{free} = 0 \quad \to \quad \epsilon\frac{\partial V_{in}}{\partial n} = \epsilon_0\frac{\partial V_{out}}{\partial n}$$

At r → Large

$$\vec{E} = E_0\hat{k} \ \to V_{out} = -E_0 r\cos\vartheta$$

Remember the solution to Laplace's equation in spherical coords has the following form:

$$V(r) = \sum\left(A_l r^l + B_l r^{-l-1}\right)P_l(\cos\vartheta)$$

Applying the first B.C. inside, we reduce it to:

$$V_{in} = \sum A_l r^l\, P_l(\cos\vartheta)$$

How about outside?

$$V_{out} = -E_0 r\cos\vartheta + \sum B_l r^{-l-1}\, P_l(\cos\vartheta)$$

Which we can rewrite

$$V_{out} = -E_0 r P_1(\cos\vartheta) + \sum B_l r^{-l-1}\, P_l(\cos\vartheta)$$

Now let's look at the boundary:

$$V_{in} = V_{out}$$

$$\sum A_l R^l\, P_l(\cos\vartheta) = -E_0 R P_1(\cos\vartheta) + \sum B_l R^{-l-1}\, P_l(\cos\vartheta)$$

For l = 1

$$A_1 = -E_0 + B_1 R^{-3}$$

Everywhere else:

$$A_l = B_l R^{2l+1}$$

The other condition:

$$\epsilon \frac{\partial V_{in}}{\partial n} = \epsilon_0 \frac{\partial V_{out}}{\partial n} \qquad n \to r$$

$$\frac{\partial V_{in}}{\partial r} = \sum l A_l r^{l-1} P_l(\cos\vartheta)$$

$$\frac{\partial V_{out}}{\partial r} = -E_0 \cos\vartheta + \sum (-l-1) B_l r^{-l-2} P_l(\cos\vartheta)$$

For l=1

$$\frac{\epsilon}{\epsilon_o} l A_l = -E_0 + (-l-1) B_l R^{-3} \to \quad A_1 = -E_0 + (-2) B_1 R^{-3}$$

Everywhere else

$$\frac{\epsilon}{\epsilon_o} l A_l = (-l-1) B_l R^{-2l+1}$$

Both A_l and B_l are zero unless l=1.

$$\frac{\epsilon}{\epsilon_o} A_1 = -E_0 + (-2) B_1 R^{-3}$$

$$A_1 = -E_0 + B_1 R^{-3}$$

The solution is:

$$A_1 = -\frac{3}{\epsilon + 2} E_0 \;\; \& \;\; B_1 = \frac{\epsilon - 1}{\epsilon + 2} R^3 E_0$$

$$V_{in} = -\frac{3}{\epsilon + 2} E_0 r \cos\vartheta = -\frac{3}{\epsilon + 2} E_0 z$$

The field is just

$$\vec{E} = -\vec{\nabla} V = \frac{3}{\epsilon + 2} \overrightarrow{E_0}$$

$$W_1 = \frac{\epsilon_0}{2}\int E^2 \, d^3x \qquad W_2 = \frac{1}{2}\int \vec{D}\cdot\vec{E} \, d^3x$$

only the charges $\quad W_1 < W_2 \quad$ charges + dipoles

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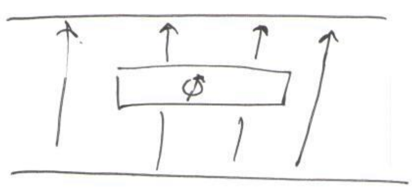
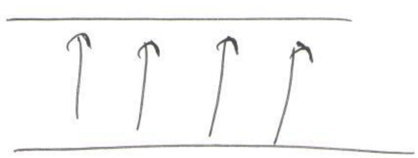
we already know that

$$\vec{F} = (\vec{p}\cdot\vec{\nabla})\vec{E}$$



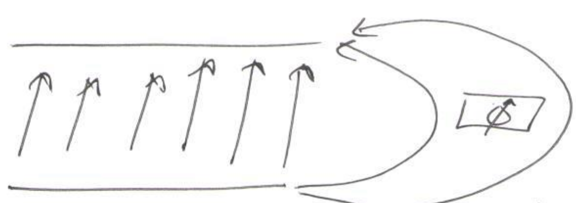
uniform Field

$\vec{F} = 0$



Far away

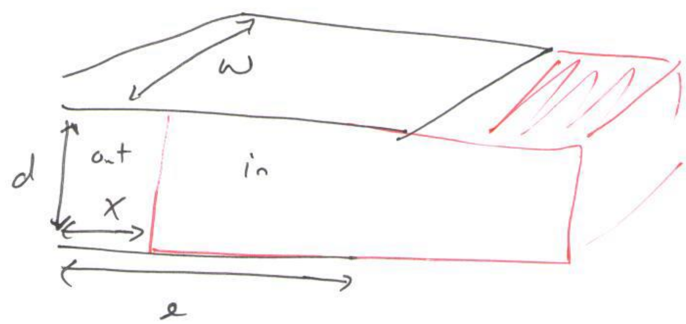
$\vec{F} = 0$



$(\vec{p}\cdot\vec{\nabla})\vec{E} \neq 0$

There's going to be a force!

So let's look at this case



Energy 1st

Capacitance should only depend on geometry and the material. here half is in and half is out.

$$C = \frac{\epsilon_0 A}{d} = \frac{\epsilon_0 A_{in}}{d} + \frac{\epsilon_0 A_{out}}{d} = \frac{\epsilon_0 w}{d} x + \frac{\epsilon w}{d}(l - x)$$

we know the following relationship

$$\epsilon_r \equiv 1 + \chi_e = \epsilon / \epsilon_0$$

$$C = \frac{\epsilon_0 w x}{d} + \frac{\epsilon_0 \epsilon_r}{d}(l - x) = \frac{\epsilon_0 w}{d}\left[x + \epsilon_r l - \epsilon_r x\right]$$

$$= \frac{\epsilon_0 w}{d}\left[\epsilon_r l - (\epsilon_r - 1)x\right] = \frac{\epsilon_0 w}{d}\left[\epsilon_r l - \chi_e x\right]$$

<u>how much Energy is stored?</u>

$$W = \sum q V \rightarrow \int_0^Q V\,dq \xrightarrow[Q = CV]{} \frac{1}{2} C V^2 \rightarrow \frac{1}{2}\frac{Q^2}{C}$$

Why Energy 1^{st}, how is the energy defined?

$$\int dW = \int \vec{F} \cdot \vec{dx}$$

- or -

$$F = -\frac{dW}{dx}$$

$$F = -\frac{d}{dx} \cdot \frac{1}{2}\frac{Q^2}{C}$$ ← NOT sure what's up with V but I know Q is a constant

$$F = -\frac{Q^2}{2}\frac{d}{dx}\frac{1}{C}$$

$$F = -\frac{Q^2}{2}\left(-\frac{1}{C^2}\frac{dC}{dx}\right)$$

$$F = \frac{Q^2}{2C^2}\frac{dC}{dx}$$

$Q = CV$

$$F = \frac{1}{2}V^2\frac{dC}{dx}$$

$$F = \frac{1}{2}V^2\frac{d}{dx}\frac{\epsilon_0 w}{d}\left[\epsilon_r l - \chi_e x\right]$$

$$F = -\frac{\epsilon_0 \chi_e w}{2d}V^2$$